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One-Dimensional Lorentz Gas with Rotating Scatterers: Exact Solutions

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The simplest solutions (orbits) to the recently introduced Lorentz gas with rotating scatterers are found by considering its one-dimensional one-particle reduction. This model has only one parameter which can be viewed as the amount of energy transfer between the scatterers and the particle during a collision. Exact solutions of the system are found for several values of this parameter. For some of these values, the dynamics is shown to be in many respects similar to the dynamics of the deterministic Lorentz lattice gases.

KEY WORDS: Lorentz lattice gas; Lorentz gas; rotating scatterers; flipping scatterers.

1. INTRODUCTION

It is well known that the classical Lorentz lattice gas does not satisfy local thermal equilibrium condition because in this model the scatterers are immovable. To overcome this deficiency a modification of the periodic Lorentz gas was introduced.[(1)] In this modification the scatterers are freely rotating disks with fixed centers. Infinitely many non-interacting point particles move in an array of such scatterers. Although the particles do not interact directly, they exchange energy with each other through the disks, by changing the disk's angular velocities. Therefore, contrary to the classical Lorentz gas, analyzing the system with just one moving particle will not give us full understanding of the dynamics of the multi-particle model.

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Nevertheless, the one-particle model is interesting by itself. It is a highly nontrivial infinite-dimensional dynamical system. The first step in studying time evolution of any dynamical system is to find its simplest solutions. This is the goal of this paper. We consider orbits in which the velocity of the particle is tangential to just one row (or column) of scatterers. In other words, we assume that the array of scatterers is one-dimensional. However, even after all these reductions are made, the dynamics is still quite complex. We have been able to analyze it completely for only a few values of the model's parameter. For some of these values, the model is equivalent to the deterministic Lorentz lattice gas (LLG) with flipping scatterers.[2–4] It is remarkable that this seemingly artificial and largely simplified models appear as an exact (albeit special) solutions to an extremely complicated model which is currently considered to be one of the most relevant systems to study the transport phenomena.

We consider a one-dimensional array of freely rotating circular scatterers of radius R with a finite moment of inertia Θ. The centers of the scatterers are fixed on a one-dimensional lattice $\mathbb{Z}$. We consider all possible initial configurations of the scatterers, i.e., all possible configurations of their angular velocities at $t = 0$. A point particle moves along the line parallel to this lattice and tangential to the disks (see Fig. 1). The normal component of the particle's velocity is assumed to be 0, so by *velocity* of the particle we mean its tangential component. For an arbitrary initial configuration of scatterers, i.e., when no assumptions are made about their distribution, the system under study becomes completely deterministic. The results obtained in this text describe the possible qualitative behaviours of its orbits. Additional probabilistic assumptions about the initial distribution of angular velocities would allow for quantitative (statistical) description of the ensemble of orbits.

Let v denote the velocity of the particle, and ω the angular velocity of a disk. The collision of the particle with the disk proceeds according to the following rules:

$$\begin{cases} v' = v - \dfrac{2\eta}{(1+\eta)}(v - R\omega) \\ \omega' = \omega + \dfrac{2}{R(1+\eta)}(v - R\omega). \end{cases} \tag{1}$$

Here v' and ω' are the velocity of the particle and the angular velocity of the disk after the collision, and $\eta = \Theta/(mR^2) > 0$ is the only parameter of the model and can be viewed as the amount of energy transfer between the disks and the particles. It should be noted that this transformation conserves energy and angular momentum.[1]

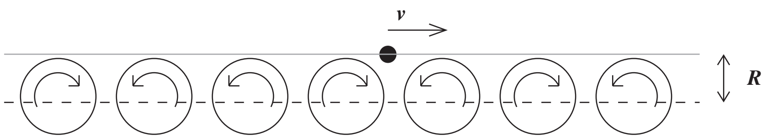

Fig. 1. The LLG with rotating scatterers under study. The centers of the scatterers are fixed on a one-dimensional lattice $\mathbb{Z}$. A point particle moves along the line parallel to this lattice and tangential to the disks. No assumptions is being made about the disk's angular velocities, i.e., they can be arbitrary.

2. EXACT SOLUTIONS

First, note that the dependence on the radius of the disks R in (1) can be eliminated by rewriting the system using disk velocities $w = R\omega$ instead of their angular velocities ω. The equations can be further simplified by introducing a new parameter

$$\kappa = \frac{\eta - 1}{\eta + 1} \in [-1, 1]$$

to replace η in (1). If $t_-(t_+)$ denotes the time instant just before (after) the interaction of the particle with the scatterer, and $t \equiv t_+$, then the system (1) is equivalent to:

$$\begin{cases} v(t+1) = v(t) - (1+\kappa)(v(t) - w(t)) \\ w(t+1) = w(t) + (1-\kappa)(v(t) - w(t)) \end{cases} \tag{2}$$

where the time is measured in the number of the particle's collisions with scatterers. We refer to this discrete time when speaking of time, unless noted otherwise. The qualitative results obtained in this text are valid for both the discrete and the real time, due to Proposition 1 below.

Since the dependence on R has been eliminated from these equations, we assume that the radii of disks in Fig. 1 are negligibly small, and the particle is moving on the lattice $\mathbb{Z}$.

This system of equations can be completely solved for some values of parameter κ. In the following subsections we describe these exact solutions. Without loss of generality we assume that the initial velocity of the particle is positive. We denote as $w_i(t)$ the angular velocity of the disk at position $i \in \mathbb{Z}$ at time t, and $w_i \equiv w_i(0)$. Throughout this paper, the rotating scatterers with positive angular velocities will be referred to as right scatterers (RS), and those with negative angular velocities as left scatterers (LS). By *flipping* a scatterer we mean changing its type, i.e., LS $\Leftrightarrow$ RS.

Proposition 1. In the one-particle LLG model (2) with $\kappa \neq -1$ the particle never stops at any vertex for more than one time step.

Proof. Suppose that the particle stops at time $(t+1)$, e.g., $v(t) \neq 0$ and $v(t+1) = 0$ for some t. Then we can solve the first equation of (2) for $w(t)$, when $\kappa \neq -1$:

$$w(t) = \frac{k}{1+k}\, v(t).$$

Now, we can use this to compute the particle's velocity at the next time instant:

$$\begin{aligned} v(t+2) &= v(t+1) - (1+\kappa)(v(t+1) - w(t+1)) = (1+\kappa)\, w(t+1) \\ &= (1+\kappa)(w(t) + (1-\kappa)(v(t) - w(t))) = v(t). \end{aligned}$$

Hence, the particle will resume its motion in the direction it was moving just before it stopped.

Remark 1. It will be shown later that, when $\kappa = -1$, the particle propagates with the constant velocity and, therefore, never stops.

Case 1: κ=0

When $\kappa = 0$ the system (2) reduces to

$$\begin{cases} v(t+1) &= w(t) \\ w(t+1) = v(t), \end{cases}$$

so the particle and the disk simply exchange velocities during the collision. If a particle moving to the right encounters a right scatterer, it passes through it and continues moving in the positive direction without flipping the scatterer. If it hits a left scatterer, however, say at position z, it bounces back to position $(z-1)$. This position, however, is occupied by a right scatterer (since the particle previously passed through it), so the particle gets reflected back to z. Note that, during the first collision of the particle with the LS at z, the scatterer flips, thus becoming RS. Hence, the second time the particle arrives at z, it passes through the scatterer without changing its direction or flipping it.

Thus, once the particle passes through a scatterer a *blocking pattern*, similar to the one in LLG models with flipping scatterers (see, e.g., ref. 5), forms behind it and prevents it from moving in the negative direction for more than one step. This proves the following

Proposition 2. In the one-particle LLG model (2) with $\kappa = 0$, the particle ultimately propagates in one direction. The direction of propagation depends on the configuration of scatterers at the particle's initial position and its nearest neighbors.

The particle with positive initial velocity ultimately propagates to the left only if $w_0 < 0$ and $w_1 < 0$. For all other combinations of scatterers it propagates to the right.

This behaviour is very similar to that of the one-dimensional *flipping mirror* model, also known as a *walk in rigid environment* with rigidity one. In fact, when the initial velocities of the disks satisfy $|w_i| = 1$ for all $i \in \mathbb{Z}$, and the disks with positive and negative velocities are distributed over the lattice vertices independently with probabilities q and $p = 1 - q$ respectively, the particle's dynamics in the two models are indentical. The average velocity of propagation in this case is related to the initial density of left scatterers p via:

$$\bar{v} = \frac{1}{1+2p}. \tag{3}$$

Case 2: $\kappa = 1$

In this case the system (2) reduces to

$$\begin{cases} v(t+1) = -v(t) + 2w(t) \\ w(t+1) = w(t), \end{cases} \tag{4}$$

so the angular velocities of the scatterers do not change in the course of evolution: $w_i(t) = w_i$; and the environment is *fixed*. We show that in this scenario there are only two possible regimes of the particle's motion: either the particle propagates in one direction on $\mathbb{Z}$, or its trajectory is periodic. Let $z(t)$ denote the particle's position at time $t \geqslant 0$.

Proposition 3

(i) If the initial configuration of scatterers in the one-particle LLG model (2) with $\kappa = 1$, satisfies:

$$2 \sum_{k=1}^{2n} (-1)^{k+1} w_{z(k)} \leqslant v(0) \leqslant 2 \sum_{k=1}^{2n-1} (-1)^{k+1} w_{z(k)} \qquad \text{for any} \quad n \geqslant 1, \tag{5}$$

then, after at most one time step the particle propagates to the right: $v(t) \geqslant 0$ for all $t \geqslant 1$.

(ii) If the initial configuration of scatterers satisfies:

$$2\sum_{k=1}^{2n-1}(-1)^{k+1}w_{z(k)} \leqslant v(0) \leqslant 2\sum_{k=1}^{2n}(-1)^{k+1}w_{z(k)} \qquad \text{for any} \quad n \geqslant 1, \tag{6}$$

then, after at most one time step, the particle propagates to the left: $v(t) \leqslant 0$ for all $t \geqslant 1$.

(iii) For any other configuration of scatterers, after a finite number of steps the particle gets trapped inside a unit interval $[i, i+1]$ for some i.

Proof. (i)–(ii) We will prove that parts (i)–(ii) of the Proposition provide the necessary and sufficient conditions for the particle's propagation. Indeed, for the particle to propagate to the right after one time step, the following inequality must be satisfied for each $t \geqslant 1$:

$$0 \leqslant v(t) = -v(t-1) + 2w_{z(t)} = (-1)^t\left(v(0) + 2\sum_{k=1}^{t}(-1)^k w_{z(k)}\right), \tag{7}$$

where $\{z(k)\}_{k\geqslant 0}$ denotes the particle's trajectory. Writing this inequality for $t = 2n$ and $t = 2n-1$ yields the pair of inequalities (5). The conditions for the particle's propagation to the left can be derived in the same manner by reversing the inequality sign in (7).

(iii) Let us suppose now that conditions (i)–(ii) are not satisfied. Since these are the necessary and sufficient conditions for the particle's propagation, there must exist such t that either $v(k) \geqslant 0$ for all $1 \leqslant k \leqslant t$ and $v(t+1) < 0$, or $v(k) \leqslant 0$ for all $1 \leqslant k \leqslant t$ and $v(t+1) > 0$. Let us assume that the former set of conditions is satisfied (the other case can be handled in the similar manner), and suppose that, at time t, the particle arrives at some position $i > 0$. Then its velocity satisfies the following equations:

$$\begin{aligned} v(t) &= -v(t-1) + 2w_i \\ v(t+1) &= -v(t) + 2w_{i+1} = v(t-1) + 2(w_{i+1} - w_i). \end{aligned} \tag{8}$$

Since, by our assumption, $v(t-1) > 0$ and $v(t+1) < 0$, the second equation of this sequence implies that the initial configuration must satisfy $w_{i+1} < w_i$.

We use induction to show that the particle will be forever trapped in the interval $[i, i+1]$. It suffices to show that the following inequalities are satisfied for all $k \geqslant 0$:

$$\begin{aligned} v(t+2k) &> 0 \\ v(t+2k+1) &< 0. \end{aligned} \tag{9}$$

Their validity for $k=0$ was shown to be a simple consequence of the conditions that constitute Case (iii). Let us assume that they are also satisfied for up to $k=n-1$. This means that during the time interval $[t, t+2n-1]$, the particle was bouncing between positions i and $i+1$. Let us evaluate the velocity of the particle at $t+2n$, noting that at this time istant it arrives at position i:

$$v(t+2n) = -v(t+2n-1)+2w_i = v(t+2n-2)-2(w_{i+1}-w_i).$$

But, as we showed earlier, $w_{i+1} < w_i$, so $v(t+2n) > v(t+2n-2) > 0$. Thus, the direction of the particle's motion changes once again, and it proceeds to position $i+1$. At the subsequent time instant we have

$$\begin{aligned} v(t+2n+1) &= -v(t+2n)+2w_{i+1} \\ &= v(t+2n-1)+2(w_{i+1}-w_i) < v(t+2n-1) < 0, \end{aligned}$$

which completes our proof by induction. The immediate corollary to inequalities (9) is that the particle turns every time it visits either i or $(i+1)$, thus never leaving the interal $[i, i+1]$. It is interesting to note that the speed of the particle is continually increasing:

$$|v(t+k)| = |v(t)|+2k(w_i - w_{i+1}), \tag{10}$$

and, therefore, tends to infinity.

Remark. The conditions of part (iii) of Proposition 3 are satisfied if, for example, $0 < v(0) < 2w_1$ and the initial concentration of left scatterers is positive. Indeed, the inequalities guarantee that this configuration does not satisfy the conditions of part (ii). Moreover, since the concentration of left scatterers is positive, then, with probability one, the particle will encounter one of them, i.e., $w_{i+1} < 0$ for some i. Then,

$$v(t+1) = -v(t)+2w_{i+1} < 0,$$

and, so this configuration does not satisfy conditions of part (i) either. Hence, it falls under conditions of part (iii).

Case 3: $\kappa = -1$

For this value of the parameter κ the system (2) reduces to

$$\begin{cases} v(t+1) = v(t) \\ w(t+1) = 2v(t) - w(t), \end{cases}$$

and the dynamics of the particle becomes trivial. The following Proposition follows immediately from these equations:

Proposition 4. In the one-particle LLG model (2) with $\kappa = -1$, the particle propagates in one direction, with constant speed $v(t) = v(0)$, for any initial configuration of scatterers.

Case 4: $-1 < \kappa < 0$

For negative values of κ, the system (2) can be rewritten in the form:

$$\begin{cases} v(t+1) = \alpha v(t) + (1-\alpha)\, w(t) \\ w(t+1) = (1+\alpha)\, v(t) - \alpha w(t), \end{cases} \tag{10}$$

where $\alpha = |\kappa|$. A simple analysis of signs of the particle's and disk's velocities, before and after the collision, allows us to prove the following:

Proposition 5. The trajectory of the particle in the one-particle LLG model (2) with $\kappa < 0$ is unbounded.

Proof. Table I lists all valid sign combinations for the velocities of the particle and the disk, before and after the collision. It is easy to verify that no other combination satisfies both equations in (11).

Suppose that the trajectory is bounded. Then the particle visits a finite number of vertices on $\mathbb{Z}$ infinitely many times. Let us consider the right-most such vertex and denote it as z^*. This means that the particle must always be scattered to the left at this vertex. But, as Table I shows, the particle moving with a positive velocity will turn back at z^* only if it

Table I. A List of Valid Sign Combinations for the Velocities of the Particle and the Disk Before and After Their Interaction, when $\kappa < 0$

$v(t)$	$w(t)$	$v(t+1)$	$w(t+1)$
+	+	+	+
+	+	+	−
+	−	+	+
+	−	−	+
−	+	+	−
−	+	−	−
−	−	−	+
−	−	−	−

encounters a left scatterer. And not only does the particle turn, it also flips the scatterer: $w(t+1) > 0$. Hence, during the particle's next visit to z^* the velocities of both participants will be positive, and the particle will pass through the scatterer. This contradicts our assumption that z^* was the right-most vertex visited by the particle. Thus, the character of interaction between the particle and scatterers pushes the particle outside of any finite region and its trajectory is unbounded. ∎

More can be said about the particle's dynamics when $\kappa = -1/2$ and the initial velocities of the particle and the disks are the same in magnitude: $|w_j(0)| = |v(0)|$ for all $j \in \mathbb{Z}$. In this case the system (11) further reduces to:

$$\begin{cases} v(t+1) = 1/2v(t) + 1/2w(t) \\ w(t+1) = 3/2v(t) - 1/2w(t), \end{cases} \tag{12}$$

and we can prove the following

Proposition 6. If the initial configuration of scatterers and the particle in the one-particle LLG model (2) with $\kappa = -1/2$, satisfy

$$|w_j(0)| = |v(0)| \qquad \text{for all} \quad j \in \mathbb{Z}$$

then the particle propagates with its initial velocity $v(0)$.

Proof. There are only two possibilities to consider: when the velocity of the particle $v(t)$ is the same as or opposite to the velocity of the scatterer $w(t)$.

If $v(t) = w(t)$ then the particle and the scatterer "ignore" each other:

$$\begin{cases} v(t+1) = v(t) \\ w(t+1) = w(t) \end{cases} \tag{13}$$

and their respective velocities do not change during the collision.

If $v(t) = -w(t)$, then the velocities of the particle and the disk at subsequent moments of time satisfy the following equations:

$$\begin{cases} v(t+1) = 0 \\ w(t+1) = -2w(t) \end{cases}$$
$$\begin{cases} v(t+2) = -w(t) = v(t) \\ w(t+2) = w(t). \end{cases} \tag{14}$$

Thus, after two time steps, the system returns to the state it was in right before the particle's collision with the scatterer. The particle stops immediately after the collision, but continues moving in its original direction after just one time step. Hence, the direction of the particle's motion never changes, and is completely determined by the direction of its initial velocity.

To compute the velocity of propagation, note that it takes one time step for the particle to pass through a right scatterer and two time steps to pass through a left scatterer. However, in real time the second time step is instantaneous since the particle is interacting twice with the same scatterer. Hence, regardless of the type of scatterer encountered by the particle, it takes the same amount of time for the particle to move one lattice step in the direction of propagation. Therefore, the particle propagates with constant speed which is independent on the initial configuration of scatterers and, according to (13)–(14), is equal to $v(0)$.

Proposition 7. If the initial configuration of scatterers and the particle in the one-particle LLG model (2) with $\kappa < 0$, satisfies

$$\operatorname{sgn} w_j = \operatorname{sgn} v(0) \qquad \text{for all} \quad j \in \mathbb{Z},$$

then the particle propagates in the direction of its initial velocity.

Proof. If the initial velocities of the disks and the particle are positive (negative), then the r.h.s. of the first equation in (11) is always positive (negative). Hence, the sign of the particle's velocity never changes in the course of propagation, so the particle propagates in the direction of its initial velocity.

Proposition 8. If the initial configuration of scatterers and the particle in the one-particle LLG model (2) with $k < 0$, satisfies both of the following conditions

(i) $v(0) > (1+1/\kappa)\, w_1$ (respectively $v(0) < (1+1/\kappa)\, w_1$)

(ii) $w_j \geqslant \kappa w_{j-1}$ (respectively $w_j \leqslant \kappa w_{j-1}$) for all $j \in \mathbb{Z}$,

then the particle propagates to the right (left).

Proof. The Proposition can be proved by induction. We consider the case of propagation to the right. The first of the two conditions guarantees that the particle will pass through the first scatterer it encounters:

$$\begin{aligned} v(1) &= -\kappa v(0) + (1+\kappa)\, w_1 \\ &= -\kappa(v(0) - (1+1/\kappa)\, w_1) > 0. \end{aligned} \tag{15}$$

Assuming that, at time i, it passes through the scatterer located at position i, i.e.:

$$v(i) = -\kappa v(i-1) + (1+\kappa)\, w_i > 0, \tag{16}$$

we evaluate the particle's velocity after its collision with the next scatterer on its path:

$$\begin{aligned} v(i+1) &= -\kappa v(i) + (1+\kappa)\, w_{i+1} \\ &= \kappa^2 v(i-1) + (1+\kappa)(w_{i+1} - \kappa w_i). \end{aligned} \tag{17}$$

Since $w_{i+1} \geqslant \kappa w_i$, the r.h.s of this equation is positive, so the velocity of the particle stays positive after the collision. Thus, the particle keeps moving in the direction.

The case of the particle propagation to the left can be considered by reversing the signs in the inequalities above.

Case 5: κ > 0

No rigorous statements concerning the particle's dynamics for positive κ can be proven as of yet. Results of simulations of this system suggest that the particle's trajectory is unbounded. At this point, however, this is only a conjecture.

We can, however, prove a statement analogous to Proposition 8 for $\kappa > 0$, by reusing Eqs. (15)–(17) without any changes:

Proposition 9. If the initial configuration of scatterers and the particle in the one-particle LLG model (2) with $\kappa > 0$, satisfies both of the following conditions

(i) $v(0) < (1+1/\kappa)\, w_1$ (respectively $v(0) > (1+1/\kappa)\, w_1$)

(ii) $w_j \geqslant \kappa w_{j-1}$ (respectively $w_j \leqslant \kappa w_{j-1}$) for all $j \in \mathbb{Z}$,

then the particle propagates to the right (left).

3. CONCLUDING REMARKS

We have found the simplest orbits of the Lorentz gas with rotating scatterers and demonstrated that, in some cases, its dynamics is similar to the dynamics of one-dimensional flipping Lorentz lattice gas.[(3)] But we also need to point out one notable difference. All of the results in this text were obtained by introducing discrete time into the model, the technique often used when studying LLG. In classical LLG cellular automata the real-time and the discrete-time dynamics are equivalent. But this is only partially the case for LLG with rotating scatterers. Qualitatively, the dynamics of the particle is the same in both the real time and the discrete time due to Proposition 1. Its quantitative characteristics, however, may be different. Indeed, the speed $|v_d|$ of the particle with respect to the discrete time is always bounded: $|v_d| \in \{0, 1\}$. Moreover, its average speed satisfies $1/2 \leqslant \langle |v_d| \rangle \leqslant 1$ (see Proposition 1). On the other hand, it is possible to find such a configuration of scatterers that the particle's speed with respect to *real time* tends to 0 or infinity (see, e.g., Eq. (9) in the proof of Proposition 6) as time increases.

The corresponding system with infinitely many particles is expected to exhibit much more complicated behaviour. Even for deterministic one-dimensional multi-particle LLG, essentially only numerical results are currently available.[6]

On the other hand, it would be interesting to compare the dynamics of the one-particle Lorentz gas with rotating scatterers on the triangular lattice[1] with the dynamics of classical LLG on this lattice.[7] One of the natural questions would be: which types of scattering rules on the latter can mimic the dynamics of the former?

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